

Functionals of Dirichlet processes, the Markov Krein Identity and Beta-Gamma processes

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This paper describes how one can use the well-known Bayesian prior to posterior analysis of the Dirichlet process, and less known results for the gamma process, to address the formidable problem of assessing the distribution of linear functionals of Dirichlet processes. In particular, in conjunction with a gamma identity, we show easily that a generalized Cauchy-Stieltjes transform of a linear functional of a Dirichlet process is equivalent to the Laplace functional of a class of, what we define as, beta-gamma processes. This represents a generalization of the Markov-Krein identity for mean functionals of Dirichlet processes. A prior to posterior analysis of beta-gamma processes is given that not only leads to an easy derivation of the Markov-Krein identity, but additionally yields new distributional identities for gamma and beta-gamma processes. These results give new explanations and interpretations of existing results in the literature. This is punctuated by establishing a simple distributional relationship between beta-gamma and Dirichlet processes.

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1 Introduction

Let P denote a Dirichlet random probability measure on a Polish space \mathcal{Y} , with law denoted as $\mathcal{D}(dP|\theta H)$, where θ is a non-negative scalar and H is a (fixed) probability measure on \mathcal{Y} . In addition let \mathcal{M} denote the space of boundedly finite measures on \mathcal{Y} . This space contains the space of probability measures on \mathcal{Y} . The Dirichlet process was first made popular in Bayesian nonparametrics by Ferguson (1973), see also Freedman (1963) for an early treatment, and has subsequently been used in numerous statistical applications. Additionally, the Dirichlet process arises in a variety of interesting contexts outside of statistics. Formally, P is said to be a Dirichlet process if and only if for each finite collection of disjoint measurable sets A_1, \dots, A_k , the random vector $P(A_1), \dots, P(A_k)$ has a Dirichlet distribution with parameters $\theta H(A_1), \dots, \theta H(A_k)$. In particular $P(A)$ is a beta random variable for any measurable set A . An important representation of the Dirichlet, which is analogous to Lukacs characterization of the gamma distribution, is

$$(1) \quad P(\cdot) = \frac{\mu(\cdot)}{T}$$

where μ is a gamma process with finite shape parameter θH and $T = \int_{\mathcal{Y}} \mu(dy)$ is a gamma random variable with shape θ and scale 1. The law of the gamma process is denoted as $\mathcal{G}(d\mu|\theta H)$, and is characterized by its Laplace functional

$$(2) \quad \int_{\mathcal{M}} e^{-\mu(g)} \mathcal{G}(d\mu|\theta H) = e^{-\int_{\mathcal{Y}} \log[1+g(y)]\theta H(dy)}$$

for g a positive measurable function satisfying,

$$(3) \quad \int_{\mathcal{Y}} \log[1+g(y)]\theta H(dy) < \infty.$$

We note that if H is non-atomic then the gamma process may be represented as

$$(4) \quad \mu(dy) = \int_0^\infty sN(ds, dy)$$

where N is a Poisson random measure with mean intensity,

$$\theta s^{-1} e^{-s} ds H(dy).$$

An important fact is that T and P are independent, which as we shall see, has a variety of implications.

An interesting and formidable problem initiated in a series of papers by Cifarelli and Regazzini(1990) is the study of the exact distribution of linear functionals,

$$P(g) = \int_{\mathcal{Y}} g(y)P(dy)$$

of the Dirichlet process. Diaconis and Kemperman (1996) discuss an important by-product of this work called the *Markov-Krein* identity for means of Dirichlet processes,

$$(5) \quad \int_{\mathcal{M}} \frac{1}{(1+zP(g))^\theta} \mathcal{D}(dP|\theta H) = e^{-\int_{\mathcal{Y}} \log[1+zg(y)]\theta H(dy)},$$

which has implications relative to the Markov moment problem, continued fractions theory, exponential representations of analytic functions, etc [see Kerov (1998) and Tsilevich, Vershik and Yor (2001a)]. Tsilevich, Vershik and Yor (2001a) expand upon this emphasizing that the right hand side of (5) is the Laplace functional of a Gamma process with shape θH . That is,

$$(6) \quad \int_{\mathcal{M}} \frac{1}{(1+zP(g))^\theta} \mathcal{D}(dP|\theta H) = \int_{\mathcal{M}} e^{-z\mu(g)} \mathcal{G}(d\mu|\theta H)$$

Their interpretation of the Markov-Krein identity is that the generalized Cauchy-Stieltjes transform, of order θ , of $P(g)$, where P is a Dirichlet process with shape θH , is the Laplace transform of $\mu(g)$ when μ is the gamma process with shape θH . The authors then exploit this fact to rederive (6) via an elementary proof using the independence property of P and T . An interesting question, is what can one say about

$$(7) \quad \int_{\mathcal{M}} \frac{1}{(1 + zP(g))^q} \mathcal{D}(dP|\theta H)$$

when θ and q are arbitrary positive numbers? That is, can one establish a relationship of (7) to the Laplace functional of some random measure say μ^* , which is similar to μ , for all q and θ . Lijoi and Regazzini (2003) establish analytic results for (7), relating them to the Lauricella theory of multiple hypergeometric functions. Theorem 5.2 of their work gives analogues of (5), stating what they call a *Lauricella identity*, but does not specifically state a relationship such as (6). We should say for the case $\theta > q$ that it would not be terribly difficult to deduce an analogue of (7) from their result. However, this is not the case when $\theta < q$, which is expressed in terms of contour integrals. Their representations, for all θ and q , as clearly demonstrated by the authors, however have practical utility in regards to formulae for the density of $P(g)$. In this case, one wants to have an expression for (7), when $q = 1$ and for all θ . Related works include the papers of Regazzini, Guglielmi and di Nunno (2002), Regazzini, Lijoi and Pruenster (2003) and the manuscript of James (2002).

1.1 Preliminaries and outline

In this paper we develop results that are complementary to the work of Lijoi and Regazzini (2003) and Tsilevich, Vershik and Yor (2001a). In particular, we show that the Markov-Krein identity, as interpreted in Tsilevich, Vershik and Yor (2001a), extends to a relationship between (7) and the Laplace functional of a class of what we call beta-gamma processes defined by scaling the gamma process law by $T^{-(\theta-q)}$, for all positive θ and q . That is processes with laws equal to

$$(8) \quad \mathcal{BG}(d\mu|\theta H, \theta - q) = \frac{\Gamma(\theta)}{\Gamma(q)} T^{-(\theta-q)} \mathcal{G}(d\mu|\theta H)$$

Perhaps more interesting, is the method of approach, and derivation of supporting results, used to establishing such a result, which is quite different than the analytic techniques used previously. The approach relies in part on, in this case mostly familiar, Bayesian prior posterior calculus for Dirichlet and gamma processes in conjunction with the usage of the following well-known *gamma identity* for $q > 0$

$$(9) \quad T^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty v^{q-1} e^{-vT} dv.$$

That is to say, purely analytic arguments are replaced by probabilistic ones using the familiar results in Ferguson (1973), Lo (1984) and Antoniak (1974). Thus giving the derivations a much more interpretable Bayesian flavor. More specifically, albeit less well known, we use the results in Lo and Weng (1989) as demonstrated for more general processes in James (2002, 2003). This bypasses the need for instance to verify certain integrability conditions and the usage of limiting arguments. Moreover, somewhat conversely to Lijoi and Regazzini (2003), we show how properties of the Dirichlet and beta-gamma processes yields easily interesting identities related to Lauricella and Liouville integrals[see Lijoi and Regazzini (2003) and Gupta and Richards (2001)]. Although we exploit the independence property of T and P to prove our results, our approach is quite different from the methods used by Tsilevich, Vershik and Yor (2001) to prove (6). While their proof is certainly elegant, it does not seem possible to extend to other processes. Our methods however are influenced by their proof of an analogous result for the two-parameter extension of the Dirichlet process which relies on (9) and the fact that such processes are based on scaled laws. That is to say we present an approach which is extendable to other models[see James (2002, section 6)].

The study of scaled laws are of clear interest in the case of the stable law of index $0 < \alpha < 1$ as discussed in Pitman and Yor (1997)[see also Pitman (2002)]. In particular the two-parameter (α, θ) extension of the Dirichlet process[see Pitman (1996)] can be defined as P in (1) which has law governed by $T^{-\theta}\mathcal{P}(d\mu|\rho_\alpha, H)$, where $\mathcal{P}(d\mu|\rho_\alpha, H)$ is the law of the stable law process, which can be derived from a Poisson random measure with intensity,

$$\rho_\alpha(ds)H(dx) = s^{-\alpha-1}dsH(dx).$$

However, for the beta-gamma processes defined in (8), the independence property between T and P translates into the following property

$$(10) \quad \int_{\mathcal{M}} f(P)\mathcal{D}(dP|\theta H) = \int_{\mathcal{M}} f(P)\mathcal{G}(d\mu|\theta H) = \int_{\mathcal{M}} f(P)\mathcal{BG}(d\mu|\theta H, \theta - q).$$

for all integrable f . The property (10) seems to suggest that the beta-gamma process may not have much utility relative to calculations involving P , however it is precisely this property that we shall exploit. The outline of this paper is as follows; in section 2 we revisit well-known properties of the Dirichlet process P and describe the posterior distribution of μ given $\mathbf{Y} = \{Y_1, \dots, Y_n\}$ when $\mathbf{Y}|P$ is an iid sample from P . In particular, we show that the posterior distribution of $\mu|\mathbf{Y}$ is a posterior beta-gamma process, and derive its Laplace functional. In section 3 we present a general analysis of the properties of the beta-gamma process including a description of its posterior distribution which is shown to be a conjugate family. Additionally formulae for the Laplace functional are derived and some key results which are relevant to the proof of the Markov-Krein identity are given. In sections 4 and 5 we present results that describes formally the various relationships between (7) and the beta-gamma processes.

2 Some results for the Dirichlet and Gamma process

We first recall some key features of the gamma and Dirichlet processes. Hereafter, we will assume that H is non-atomic, and that each function g satisfies (3). Let Y_1, \dots, Y_n denote random elements in the space \mathcal{Y} , which conditional on P are iid with law P . P is a Dirichlet process with shape θH . From Lo and Weng (1989)[see also James (2002, 2003)], one has the following disintegration of measures,

$$(11) \quad \prod_{i=1}^n \mu(dY_i)\mathcal{G}(d\mu|\theta H) = \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})\theta^{n(\mathbf{p})} \prod_{j=1}^{n(\mathbf{p})} \Gamma(e_{j,n})H(dY_j^*),$$

where $\mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$ denotes a gamma process with shape $\theta H + \sum_{i=1}^n \delta_{Y_i}$. The quantity $\mathbf{p} = \{C_1, \dots, C_{n(\mathbf{p})}\}$ denotes a partition of the integers $\{1, \dots, n\}$, with $n(\mathbf{p})$ elements. The $C_j = \{i : Y_i = Y_j^*\}$ for $j = 1, \dots, n(\mathbf{p})$ denote the collection of values equal to each unique Y_j^* , for $j = 1, \dots, n(\mathbf{p})$. The size of each cell C_j is denoted as $e_{j,n}$. Recall that a gamma process can also be described as in (4) in terms of a Poisson random measure with mean intensity (1). Another important property of the Gamma process is the following exponential change of measure formula,

$$(12) \quad e^{-\mu(g)}\mathcal{G}(d\mu|\theta H) = \mathcal{G}(d\mu|\rho_{1+g}, \theta H)e^{-\int_{\mathcal{Y}} \log[1+g(y)]\theta H(dy)}$$

where $\mathcal{G}(d\mu|\rho_{1+g}, \theta H)$ denotes a weighted gamma process defined for each Borel measurable set A as,

$$(13) \quad \int_A [1 + g(y)]^{-1} \mu(dy)$$

or equivalently by its inhomogeneous Lévy measure $\rho_{1+g}(ds|y) = e^{-(1+g(y))s} s^{-1} ds$.

REMARK 1. The result (12) appears in Lo and Weng (1989, Proposition 3.1) and has been independently derived in Tsilevich, Vershik and Yor (2001b). Versions of this result also hold for the case where g is negative or complex valued. For this study, we can actually bypass the explicit usage of (12) but use it to demonstrate how one might apply a similar result for more general random processes as in James (2002). In particular, we note that this operation is useful for the stable law, which does not admit a result similar to (11)

As an illustration, similar to Lo and Weng (1989, Corollary 3.1), we show how one uses (11) to obtain the classic results for the Dirichlet process via its representation as a normalized gamma process. In particular, we show how one easily establishes the disintegration,

$$(14) \quad \prod_{i=1}^n P(dY_i) \mathcal{D}(dP|\theta H) = \mathcal{D}(dP|\theta H + \sum_{j=1}^n \delta_{Y_j}) \pi(\mathbf{p}|\theta) \prod_{j=1}^{n(\mathbf{p})} H(dY_j^*),$$

where $\mathcal{D}(dP|\theta H + \sum_{j=1}^n \delta_{Y_j})$ is a Dirichlet process with shape $\theta H + \sum_{j=1}^n \delta_{Y_j}$, and is the posterior distribution of $P|\mathbf{Y}$ [see Ferguson (1973)]. The partition probability

$$\pi(\mathbf{p}|\theta) = \frac{\Gamma(\theta)}{\Gamma(\theta + n)} \theta^{n(\mathbf{p})} \prod_{j=1}^{n(\mathbf{p})} \Gamma(e_{j,n}),$$

is a variant of Ewens sampling formula derived by Ewens (1972) and Antoniak (1974). The marginal distribution of $\{Y_1, \dots, Y_n\}$ is the Blackwell-MacQueen Pólya urn distribution which can be represented as

$$(15) \quad \mathbb{P}(d\mathbf{Y}|\theta H) = \pi(\mathbf{p}|\theta) \prod_{j=1}^{n(\mathbf{p})} H(dY_j^*).$$

The result in (14) can be obtained by working with the joint probability measure of \mathbf{Y} and μ given by replacing $\mathcal{D}(dP|\theta H)$ on the left hand side of (14) with the gamma process law, $\mathcal{G}(d\mu|\theta H)$. The task then shifts to finding the disintegration of the law of $\{\mathbf{Y}, \mu\}$ in terms of the posterior distribution of $\mu|\mathbf{Y}$ and the marginal distribution of \mathbf{Y} . An application of (11) shows that the joint distribution of $\{\mathbf{Y}, \mu\}$ has the following disintegrations,

$$(16) \quad T^{-n} \prod_{i=1}^n \mu(dY_i) \mathcal{G}(d\mu|\theta H) = T^{-n} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}) \theta^{n(\mathbf{p})} \prod_{j=1}^{n(\mathbf{p})} \Gamma(e_{j,n}) H(dY_j^*).$$

First it is clear that the marginal distribution of \mathbf{Y} is obtained by integrating out T^{-n} with respect to $\mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$. Under this law, T is a gamma random variable with shape parameter $\theta + n$ and unit scale. This implies that,

$$\int_{\mathcal{M}} T^{-n} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}) = \frac{\Gamma(\theta)}{\Gamma(\theta + n)},$$

yielding the desired expression for the marginal distribution of \mathbf{Y} in (15). Now in order to obtain the posterior distribution of P given \mathbf{Y} , first notice that the posterior distribution of μ given \mathbf{Y} is obtained by dividing (16) by the marginal distribution of \mathbf{Y} . That is, we see that its posterior distribution is of the form in (8) with specific law

$$(17) \quad \mathcal{B}\mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}, n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} T^{-n} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$$

It is not immediately obvious that the posterior distribution of $\mu|\mathbf{Y}$ in (17) indicates that the posterior distribution $P|\mathbf{Y}$ is $\mathcal{D}(dP|\theta H + \sum_{i=1}^n \delta_{Y_i})$. However, note that subject to the gamma process law $\mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$, P has the desired Dirichlet process distribution and is independent of T . Hence for any integrable f , using the independence of P and T , it follows from (10) that the posterior distribution of P given \mathbf{Y} is characterized by

$$\int_{\mathcal{M}} f(P) \pi(d\mu|\mathbf{Y}) = \int_{\mathcal{M}} f(P) \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}) = \int_{\mathcal{M}} f(P) \mathcal{D}(dP|\theta H + \sum_{i=1}^n \delta_{Y_i})$$

for f arbitrary integrable functions, which yields the desired result.

2.1 The posterior distribution of the gamma process

While the posterior distribution of P given \mathbf{Y} is well-known to be a Dirichlet process, the corresponding posterior distribution of μ given \mathbf{Y} described is not a Gamma process. In fact it is no longer a Lévy process. In order to understand what this distribution is we find it quite useful to employ the gamma identity (9). In general, the gamma identity allows one to employ an exponential change of measure formula such as (12) to handle terms like T^{-q} . For the present setting note that if T itself is a gamma random variable with shape parameter τ and unit scale, then,

$$E[e^{-vT}] = (1+v)^{-\tau}.$$

Now if $\tau - q > 0$, Fubini's theorem yields,

$$E[T^{-q}] = \frac{1}{\Gamma(q)} \int_0^\infty v^{q-1} (1+v)^{-\tau} dv = \frac{\Gamma(\tau-q)}{\Gamma(\tau)}$$

This leads to the identification of a *gamma-gamma* density with parameters τ and q denoted as,

$$(18) \quad \gamma(dv|\tau, q) = \frac{\Gamma(\tau)}{\Gamma(q)\Gamma(\tau-q)} v^{q-1} (1+v)^{-\tau} dv \text{ for } 0 < v < \infty.$$

Using the transformation $u = 1/(1+v)$ yields the density of a beta random variable with parameters $\tau - q$ and q denoted as

$$(19) \quad \mathcal{B}(du|\tau - q, q) = \frac{\Gamma(\tau)}{\Gamma(q)\Gamma(\tau-q)} u^{\tau-q-1} (1-u)^{q-1} \text{ for } 0 < u < 1.$$

In this section we will encounter the special case of $\tau = \theta + n$ and $q = n$, yielding the densities $\gamma(dv|\theta + n, n)$ and

$$\mathcal{B}(du|\theta, n) = \frac{\Gamma(\theta + n)}{\Gamma(n)\Gamma(\theta)} u^{\theta-1} (1-u)^{n-1} du \text{ for } 0 < u < 1$$

The gamma identity (9), (12) and Fubini's theorem yields the following explicit description of the posterior law of $\mu|\mathbf{Y}$,

Proposition 2.1 *Let μ denote a gamma process with law $\mathcal{G}(d\mu|\theta H)$ and let $P(\cdot) = T^{-1}\mu(\cdot)$, where $T = \mu(\mathcal{Y})$, denote a Dirichlet process with shape θH . Suppose that $Y_1, \dots, Y_n|P$ are iid P , then the posterior distribution of $\mu|\mathbf{Y}$ is $\mathcal{BG}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}, n)$ as defined in (17). Its Laplace functional is expressible as*

$$(20) \quad \int_0^\infty e^{-\int_{\mathcal{Y}} \log[1 + \frac{1}{1+v} g(y)] \theta H(dy)} \prod_{j=1}^{n(\mathbf{p})} \left(1 + \frac{1}{1+v} g(Y_j^*)\right)^{-e_{j,n}} \gamma(dv|\theta + n, n)$$

The Laplace functional can also be represented in terms of a beta density as,

$$(21) \quad \int_0^1 e^{-\int_{\mathcal{Y}} \log[1+ug(y)]\theta H(dy)} \prod_{j=1}^{n(\mathbf{p})} (1+ug(Y_j^*))^{-e_{j,n}} \mathcal{B}(du|\theta, n)$$

The Laplace functional in (21) shows that the posterior distribution of $\mu|\mathbf{Y}$ is equivalent to the distribution of the random measure $U_n G_n^*$, where U_n is a beta random variable with parameters (θ, n) and independent of U_n , G_n^* is a gamma process with shape $\theta H + \sum_{i=1}^n \delta_{Y_i}$. As shown by (20), $U_n = 1/(1+V_n)$, where V_n is a gamma-gamma random variable with parameters $(\theta+n, n)$.

PROOF. First using (9), one can write

$$(22) \quad \Gamma(n) \int_{\mathcal{M}} e^{-\mu(g)} T^{-n} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}) = \int_0^\infty v^{n-1} \int_{\mathcal{M}} e^{-\mu(g)} e^{-vT} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}) dv$$

Recalling that the exponential change of measure (12) transforms a gamma process to a weighted gamma process, two applications show that $e^{-\mu(g)} e^{-vT} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$ is equivalent to,

$$(23) \quad \mathcal{G}(d\mu|\rho_{[1+v+g]}, \theta H + \sum_{i=1}^n \delta_{Y_i}) e^{-\int_{\mathcal{Y}} \log[1+\frac{1}{1+v}g(y)]\theta H(dy)} \prod_{j=1}^{n(\mathbf{p})} (1+\frac{1}{1+v}g(Y_j^*))^{-e_{j,n}} (1+v)^{-\theta+n}$$

Now substituting the quantities depending on μ in the right hand side of (22) with (23) yields the first expression for the Laplace functional. The second expression is obtained by the change of variable $u = 1/(1+v)$ \square

We can now use the explicit description of the posterior distribution of $\mu|\mathbf{Y}$ to deduce the posterior distribution of $P|\mathbf{Y}$. Note that subject to the posterior law of $\mu|\mathbf{Y}$, the posterior distribution of $P|\mathbf{Y}$ must be equivalent to the law of the process

$$P_n^*(\cdot) = \frac{U_n G_n^*(\cdot)}{U_n G_n^*(\mathcal{Y})} = \frac{G_n^*(\cdot)}{G_n^*(\mathcal{Y})}$$

which shows, as desired, that P_n^* is a Dirichlet process with shape $\alpha H + \sum_{i=1}^n \delta_{X_i}$.

REMARK 2. Although stated a bit differently, Proposition 2.1 is equivalent to Proposition 5.7 of James (2002).

3 Properties of Beta-Gamma processes

As seen, the posterior distribution of $\mu|\mathbf{Y}$ is a special case of a class of mixed gamma processes defined by a scaling operation in (8) and called beta-gamma processes. For shorthand, we say that μ is $\mathcal{BG}(\theta H, \theta - q)$, to indicate that μ has a beta-gamma law defined by $\mathcal{BG}(d\mu|\theta H, \theta - q)$. When $\theta - q = 0$, the process reduces to a gamma process with shape θH . Similar to the proof of Proposition 2.1, in the case that $\theta - q > 0$, one may use (9) and (12) to obtain the Laplace functional given by the following proposition.

Proposition 3.1 *Let μ denote a beta-gamma process with law $\mathcal{BG}(d\mu|\theta H, \theta - q)$ such that $\theta - q > 0$ then the Laplace functional is given by*

$$(24) \quad \int_{\mathcal{M}} e^{-\mu(g)} \mathcal{BG}(d\mu|\theta H, \theta - q) = \int_0^1 e^{-\int_{\mathcal{Y}} \log[1+ug(y)]\theta H(dy)} \mathcal{B}(du|q, \theta - q).$$

Using the transformation $v = u/(1-u)$ the Laplace functional can also be written in terms of a gamma-gamma density with parameters $(\theta - q, \theta)$.

Next we describe the posterior distribution of the general class of beta-gamma processes.

Proposition 3.2 *Let μ denote a beta-gamma process with law $\mathcal{BG}(d\mu|\theta H, \theta - q)$ and let $P(\cdot) = T^{-1}\mu(\cdot)$, where $T = \mu(\mathcal{Y})$, denote a Dirichlet process with shape θH . Suppose that $Y_1, \dots, Y_n|P$ are iid P , then the posterior distribution of $\mu|\mathbf{Y}$ is $\mathcal{BG}(\theta H + \sum_{i=1}^n \delta_{Y_i}, \theta + n - q)$, defined by the law*

$$(25) \quad \mathcal{BG}(\theta H + \sum_{i=1}^n \delta_{Y_i}, \theta + n - q) = \frac{\Gamma(\theta + n)}{\Gamma(q)} T^{-(\theta+n-q)} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$$

If $\theta + n - q > 0$, then the Laplace functional of the posterior distribution is

$$(26) \quad \int_0^1 e^{-\int_{\mathcal{Y}} \log[1+ug(y)]\theta H(dy)} \prod_{j=1}^{n(\mathbf{p})} (1 + ug(Y_j^*))^{-e_{j,n}} \mathcal{B}(du|q, \theta + n - q).$$

PROOF. Note that the marginal distribution of \mathbf{Y} is the Blackwell-MacQueen urn distribution. The posterior distribution can be obtained easily by following the arguments for the case of the gamma process. In particular one can work with $T^{-(\theta+n-q)}$ in place of T^{-n} in (16) and adjust for normalizing constants. If $\theta + n - q > 0$, then the Laplace functional is obtained by using the proof of Proposition 2.1 with $\Gamma(n)$ and v^{n-1} in (22) replaced by $\Gamma(\theta + n - q)$ and $v^{\theta+n-q-1}$. \square

Using standard Bayesian arguments, we obtain an identity for the Laplace functional of a gamma process, and also an expression of the Laplace functional for all beta-gamma processes.

Proposition 3.3 *Let μ be a beta-gamma process, $\mathcal{BG}(\theta H, \theta - q)$ then choosing an integer $n \geq 0$ such that $\theta + n - q > 0$, the Laplace functional, $\int_{\mathcal{M}} e^{-\mu(g)} \mathcal{BG}(d\mu|\theta H, \theta - q)$, can be expressed as follows,*

$$\sum_{\mathbf{p}} \pi(\mathbf{p}|\theta) \int_0^1 e^{-\int_{\mathcal{Y}} \log[1+ug(y)]\theta H(dy)} \left[\prod_{j=1}^{n(\mathbf{p})} \int_{\mathcal{Y}} (1 + ug(y))^{-e_{j,n}} H(dy) \right] \mathcal{B}(du|q, \theta + n - q)$$

for all positive θ and q . Using a change of variable $v = u/(1 - u)$ leads to an expression in terms of a gamma-gamma density.

PROOF. The proof follows from standard Bayesian theory and the expression in (26). That is, the Laplace functional can be obtained by integrating the Laplace functional of the posterior distribution described in (26) with respect the Blackwell-MacQueen distribution as follows,

$$(27) \quad \int_{\mathcal{Y}^n} \left[\int_{\mathcal{M}} e^{-\mu(g)} \mathcal{BG}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}, \theta + n - q) \right] \mathbb{P}(d\mathbf{Y}|\theta H)$$

\square

We close this section with another interesting result which will be used in the coming section.

Proposition 3.4 *Let θ, q be arbitrary non-negative scalars. Then for any integer $n \geq 0$ that satisfies the constraint, $\theta + n - q > 0$ the following formula holds,*

$$(28) \quad \int_{\mathcal{M}} \frac{1}{(T + \mu(g))^q} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}) = \frac{\Gamma(\theta + n - q)}{\Gamma(q)} \int_{\mathcal{M}} e^{-\mu(g)} T^{-(\theta+n-q)} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$$

The expressions are equivalent to

$$(29) \quad \frac{\Gamma(\theta + n - q)}{\Gamma(\theta + n)} \int_{\mathcal{M}} e^{-\mu(g)} \mathcal{BG}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}, \theta + n - q),$$

whose explicit expression is deduced from (26).

PROOF. From (29) and (26), the proof proceeds by showing that,

$$(30) \quad \frac{\Gamma(\theta + n)}{\Gamma(\theta + n - q)} \int_{\mathcal{M}} \frac{1}{(T + \mu(g))^q} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$$

is equivalent to (26). Apply the gamma identity to $(T + \mu(g))^{-q}$ and then two applications of (12) to $e^{-v[\mu(g)+T]} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i})$ to show that (30) is equal to,

$$\frac{\Gamma(\theta + n)}{\Gamma(\theta + n - q)\Gamma(q)} \int_0^\infty v^{q-1} e^{-\int_{\mathcal{Y}} \log[1 + \frac{v}{1+v} g(y)] \theta H(dy)} \prod_{j=1}^{n(\mathbf{p})} \left(1 + \frac{v}{(1+v)} g(Y_j^*)\right)^{-e_{j,n}} (1+v)^{-\theta+n} dv.$$

The result is obtained by applying the transformation $u = v/(1+v)$. \square

4 Functionals of Dirichlet processes, The Markov-Krein Identity and Beta-Gamma processes

Here we show that results relating (7) to the beta-gamma process can be deduced from (9), (12) and the the posterior distribution of μ . A general strategy is formed by first writing

$$\frac{1}{1 + zP(g)} = \frac{T}{T + z\mu(g)}$$

Before proceeding to the main result we illustrate the idea for the special case of $\theta - q > 0$. First write,

$$(31) \quad \int_{\mathcal{M}} \frac{1}{(1 + zP(g))^q} \mathcal{D}(dP|\theta H) = \int_{\mathcal{M}} \frac{T^q}{(T + z\mu(g))^q} \mathcal{G}(d\mu|\theta H)$$

Note the presence of T^q causes some difficulties. However when $\theta - q > 0$ we can use directly the special relationship between the Dirichlet process and the beta-gamma processes exhibited in (10). That is replacing $\mathcal{G}(d\mu|\theta H)$ with $\mathcal{BG}(d\mu|\theta H, q)$ in (31) yields,

$$(32) \quad \int_{\mathcal{M}} \frac{1}{(1 + zP(g))^q} \mathcal{D}(dP|\theta H) = \frac{\Gamma(\theta)}{\Gamma(\theta - q)} \int_{\mathcal{M}} \frac{1}{(1 + z\mu(g))^q} \mathcal{G}(d\mu|\theta H)$$

At this point one can evaluate the expression (32) using (9). However a direct appeal to Proposition 3.4 with $n = 0$ shows that

$$\int_{\mathcal{M}} \frac{1}{(1 + zP(g))^q} \mathcal{D}(dP|\theta H) = \frac{\Gamma(\theta)}{\Gamma(q)} \int_{\mathcal{M}} e^{-\mu(g)} T^{-(\theta-q)} \mathcal{G}(d\mu|\theta H) = \int_{\mathcal{M}} e^{-\mu(g)} \mathcal{B}(d\mu|\theta H, \theta - q)$$

The case where q is arbitrary requires another strategy that involves (11) and the posterior distribution of $\mu|\mathbf{Y}$. Using these arguments we present a new result which relates the generalized Cauchy-Stieltjes transform of Dirichlet process linear functionals to the Laplace functional of beta-gamma processes. This presents a generalization of the Markov-Krein identity, complementary to the Lauricella identities deduced in Lijoi and Regazzini (2003, Theorem 5.2). We also present some interesting additional identities.

Theorem 4.1 Let $\mathcal{D}(dP|\theta H)$ denote a Dirichlet process with shape θH . Let g denote a function satisfying (3), then the following relationships are established,

(i) for any positive q and θ ,

$$(33) \quad \int_{\mathcal{M}} (1 + zP(g))^{-q} \mathcal{D}(dP|\theta H) = \int_{\mathcal{M}} e^{-z\mu(g)} \mathcal{BG}(d\mu|\theta H, \theta - q).$$

Statement (i) implies the following results.

(ii) For any positive q and θ , and any integer $n \geq 0$ which satisfies $\theta + n - q > 0$, the quantities in (33) are equivalent to,

$$\sum_{\mathbf{p}} \pi(\mathbf{p}|\theta) \int_0^1 e^{-\int_{\mathcal{Y}} \log[1+ug(y)]\theta H(dy)} \left[\prod_{j=1}^{n(\mathbf{p})} \int_{\mathcal{Y}} (1 + ug(y))^{-e_{j,n}} H(dy) \right] \mathcal{B}(du|q, \theta + n - q),$$

for the gamma process, its Laplace functional may be represented as above for all $n \geq 1$ and $q = \theta$.

(iii) When, $\theta - q > 0$ statement (i) combined with Lemma 3.1 imply that

$$(34) \quad \int_{\mathcal{M}} (1 + zP(g))^{-q} \mathcal{D}(dP|\theta H) = \int_0^1 e^{-\int_{\mathcal{Y}} \log[1+ug(y)]\theta H(dy)} \mathcal{B}(du|q, \theta - q),$$

which coincides with the result in Lijoi and Regazzini (2003, Theorem 5, equation (5.2)).

PROOF. For the proof of statement (i), we first assume without loss of generality that $q = n + d$, where d is a positive scalar such that $\theta - d > 0$, and $n \geq 0$ is an integer chosen such that $\theta + n - q > 0$. This means that $T^q = T^{n+d}$. Now using (10) with $\mathcal{BG}(d\mu|\theta H, d)$ yields the expression,

$$(35) \quad \int_{\mathcal{M}} (1 + zP(g))^{-q} \mathcal{D}(dP|\theta H) = \frac{\Gamma(\theta)}{\Gamma(\theta - d)} \int_{\mathcal{M}} \frac{T^n}{(T + z\mu(g))^q} \mathcal{G}(d\mu|\theta H).$$

Now to handle the term T^n , write it as $T^n = \int_{\mathcal{Y}^n} \prod_{i=1}^n \mu(dY_i)$, then apply (11) to show that the expressions in (35) are equal to,

$$\frac{\Gamma(\theta + n)}{\Gamma(\theta - d)} \int_{\mathcal{Y}^n} \left[\int_{\mathcal{M}} \frac{1}{(T + z\mu(g))^q} \mathcal{G}(d\mu|\theta H + \sum_{i=1}^n \delta_{Y_i}) \right] \mathbb{P}(d\mathbf{Y}|\theta H)$$

Apply Proposition 3.4 to the inner term, recalling that $\theta + n - q = \theta - d$. This yields the desired expression as in (27). \square

We now discuss some interesting results obtained from Theorem 4.1. Let $\mathcal{L}(Z)$ denote the law of a random element Z . Recall that for a Dirichlet process functional, $P(g)$, based on a Dirichlet process with shape θH , one can represent its distribution as follows

$$(36) \quad \mathcal{L}(P(g)) = \mathcal{L}(U_{\theta,1}P(g) + (1 - U_{\theta,1})g(Y))$$

where P is $\mathcal{D}(\theta H)$, $U_{\theta,1}$ is a beta random variable with parameters $(\theta, 1)$ and Y has distribution H . Additionally P, U, Y are independent. The right hand side of (36) just follows from the fact that for fixed Y , it is the posterior distribution of $P|Y$ based on $n = 1$ observation. Hence the unconditional distribution must be $\mathcal{D}(\theta H)$. That is,

$$\mathcal{D}(dP|\theta H) = \int_{\mathcal{Y}} \mathcal{D}(dP|\theta H + \delta_y) H(dy).$$

See Diaconis and Freedman (1999) for an interesting usage of (36). Here we point out some related results for μ . In particular, as pointed out in Lijoi and Regazzini (2003), for determining the density of functionals $P(g)$, it is useful to have the expression for the Cauchy-Stieltjes transform for the case $q = 1$. Theorem 4.1 shows that

$$(37) \quad \int_{\mathcal{M}} (1 + zP(g))^{-1} \mathcal{D}(dP|\theta H) = \int_{\mathcal{M}} e^{-z\mu(g)} \mathcal{BG}(d\mu|\theta H, \theta - 1).$$

Thus in principle one can use the right hand side of (37) to perform an appropriate inversion to deduce the distribution of $P(g)$. See Lijoi and Regazzini (2003, Section 6) for details in that direction. Here, similar to the case of (36) we use the explicit posterior analysis of μ to characterize beta-gamma processes with shape θH and $q = 1$ for all θ . For the remainder of this work, let $\mu_{\theta, \theta-1}$ be $\mathcal{BG}(\theta H, \theta - 1)$, $U_{a,b}$ denote a beta (a, b) random variable, let T_p denote a gamma random variable with shape p and scale 1, and let Y_1 be a random element with distribution H . Additionally, let μ_θ denote a gamma process with shape θH and assume that the variables $\mu_\theta, U_{a,b}, T_p, Y_1$ are independent. Additionally let P_θ denote a Dirichlet process with shape θH .

Proposition 4.1 *Let $\mu_{\theta, \theta-1}$ be $\mathcal{BG}(\theta H, \theta - 1)$ then for g a real valued function such that its absolute value satisfies (3), the following distributional equalities hold;*

(i) for all $\theta > 0$

$$(38) \quad \mathcal{L}(\mu_{\theta, \theta-1}(g)) = \mathcal{L}(U_{1, \theta} \mu_\theta(g) + U_{1, \theta} T_1 g(Y_1))$$

(ii) for $\theta > 1$,

$$\mathcal{L}(U_{1, \theta-1} \mu_\theta(g)) = \mathcal{L}(U_{1, \theta} \mu_\theta(g) + U_{1, \theta} T_1 g(Y_1))$$

(iii) For $\theta = 1$, $\mu_{1,0} := \mu_1$ is a gamma process with shape H and

$$\mathcal{L}(\mu_1(g)) = \mathcal{L}(U_{1,1} \mu_1(g) + U_{1,1} T_1 g(Y_1))$$

(iv) If μ_θ denotes a gamma process with arbitrary shape parameter θH then,

$$(39) \quad \mathcal{L}(\mu_\theta(g)) = \mathcal{L}(U_{\theta,1} \mu_\theta(g) + U_{\theta,1} T_1 g(Y_1))$$

PROOF. The proof is immediate from statement (ii) of Theorem 4.1 or Proposition 3.3. One may also use Proposition 2.1 and Proposition 3.1 to obtain statement (iv) and (ii). It is well-known that the condition (3) is necessary and sufficient for the existence of $P(g)$. \square

The equation (38) provides a simple description of the law associated with the right hand side of (37) and should prove particularly useful in the case where $0 < \theta < 1$. Notice that the only unknown quantity in (38) is the distribution of $\mu_\theta(g)$. This suggests that in principle one can concentrate on the law of $\mu_\theta(g)$ to ascertain the law of $P_\theta(g)$ for all θ . Note this is not an obvious fact as one can use the representation

$$(40) \quad T_\theta P_\theta(g) = \mu_\theta(g),$$

and the independence of T_θ and P_θ , to find easily the distribution of μ_θ . However if one has the distribution of $\mu_\theta(g)$ one has to negotiate the typically complex dependence structure of T_θ and μ_θ to obtain the law of $P_\theta(g)$

One may use (37) and (38) to recover the expressions in Propositions 4 and 5 of Regazzini, Guglielmi and Di Nunno (2002) where Proposition 5 is a special case of the results in Regazzini, Lijoi and Pruenster (2003). The statement, (39) provides a distributional identity for an arbitrary gamma process which we believe is new. One may verify (37), by noting that (38) satisfies,

$$(41) \quad \mathcal{L}(\mu_{\theta, \theta-1}(g)) = \mathcal{L}(U_{1, \theta} \mu_\theta(g) + U_{1, \theta} T_1 g(Y_1)) = \mathcal{L}(T_1 P_\theta(g))$$

where $T_1 = U_{1,\theta}T_{\theta+1}$, with $U_{1,\theta}$ and $T_{\theta+1}$ independent, is independent of $P_\theta(g)$, where P_θ is $\mathcal{D}(\theta H)$. To see this, write

$$U_{1,\theta}[\mu_\theta(g) + T_1g(Y_1)] = U_{1,\theta}T_{\theta+1} \frac{\mu_\theta(g) + T_1g(Y_1)}{T_{\theta+1}}$$

and use the classic beta gamma calculus for random variables and the usual properties of Dirichlet, gamma processes. (41) combined with (37) yield the the obvious result,

$$\int_{\mathcal{M}} (1 + zP(g))^{-1} \mathcal{D}(dP|\theta H) = \int_0^\infty \int_{\mathcal{M}} e^{-ztP(g)} \mathcal{D}(dP|\theta H) e^{-t} dt$$

One may also use (39) to prove (5) and (6).

5 What is a beta-gamma process?

The final result which is a generalization of Proposition 4.1, is derived directly from Proposition 3.2 and Proposition 3.3. This result explains what a beta-gamma process is and indeed implies Theorem 4.1.

Theorem 5.1 *Let $\mu_{\theta,\theta-q}$ have distribution $\mathcal{BG}(\theta H, \theta - q)$ and let μ_θ denote a gamma process with shape θH , then for all positive θ and q and an integer n chosen such that $\theta + n - q > 0$, the following distributional equalities hold;*

(i) *For all θ and $q > 0$ and an integer n chosen such that $\theta + n - q > 0$,*

$$(42) \quad \mathcal{L}(\mu_{\theta,\theta-q}) = \mathcal{L}(U_{q,\theta+n-q}\mu_\theta + U_{q,\theta+n-q} \sum_{j=1}^{n(\mathbf{p})} G_{j,n} \delta_{Y_j^*}),$$

where conditional on \mathbf{p} the distinct variables on the right hand side are mutually independent such that, $U_{q,\theta+n-q}$ is beta with parameters $(q, \theta + n - q)$, μ_θ is a gamma process with shape θH , $\{G_{j,n}\}$ are independent gamma random variables with shape $e_{j,n}$ and scale 1, and Y_j^ for $j = 1, \dots, n(\mathbf{p})$ are iid H . The distribution of \mathbf{p} is $\pi(\mathbf{p}|\theta)$.*

(ii) *For $\theta - q > 0$, $\mathcal{L}(\mu_{\theta,\theta-q}) = \mathcal{L}(U_{q,\theta-q}\mu_\theta)$*

Statement (i) implies the following results.

(iii) *For all positive θ and q*

$$(43) \quad \mathcal{L}(\mu_{\theta,\theta-q}) = \mathcal{L}(T_q P_\theta)$$

where T_q is a gamma random variable with shape q and scale 1 independent of P_θ which is a Dirichlet process with shape θH .

(iv) *For all positive θ and q , $\mathcal{L}(T_\theta \mu_{\theta,\theta-q}) = \mathcal{L}(T_q \mu_\theta)$, where T_θ is gamma with shape θ and scale 1, independent of $\mu_{\theta,\theta-q}$. Similarly T_q and μ_θ are independent.*

PROOF. The distributional identity in (i) is a direct consequence of the mixture representation deduced from Proposition 3.2 which is deduced from the posterior distribution in Proposition 3.1. Note that all quantities on the right side of (42), including \mathbf{p} , are random. Statement (ii) follows from Proposition 3.1. We now show that statement(iii) follows from statement (i). Notice that, $T_{\theta+n} := \mu_\theta(\mathcal{Y}) + \sum_{j=1}^{n(\mathbf{p})} G_{j,n}$, is a gamma random variable with shape $\theta + n$ independent of $U_{q,\theta+n-q}$. Moreover using the mixture representation of the Dirichlet process derived from its posterior distribution and $\mathbb{P}(d\mathbf{Y}|\theta H)$, it follows that,

$$\frac{\mu_\theta + \sum_{j=1}^{n(\mathbf{p})} G_{j,n} \delta_{Y_j^*}}{T_{\theta+n}}$$

is a Dirichlet process with shape θH , independent of $T_{\theta+n}$ and $U_{q,\theta+n-q}$. Hence the right hand side of (42) can be written as,

$$U_{q,\theta+n-q}T_{\theta+n}P_{\theta}$$

The result is completed by noting that $U_{q,\theta+n-q}T_{\theta+n}$ is equal in distribution to T_q . Statement (iv) follows immediately from (iii). \square

The expression (43) tells us precisely that, for all θ and q , a beta-gamma process with parameters θH and $\theta - q$ is equivalent in distribution to a Dirichlet process with shape θH , scaled by an independent gamma random variable with shape q . Hence, using this interpretation the first result in Theorem 4.1 is an immediate consequence of,

$$E[e^{-z\mu_{\theta,\theta-q}(g)}] = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} \left[\int_{\mathcal{M}} e^{-ztP(g)} \mathcal{D}(dP|\theta H) \right] e^{-t} dt = \int_{\mathcal{M}} (1 + zP(g))^{-q} \mathcal{D}(dP|\theta H).$$

However, it is the other representations of the beta-gamma process that yield useful explicit expressions.

6. Closing Remarks. The methodology discussed here relies on three main ingredients which are not specifically linked to the Dirichlet or Gamma process. That is, the use of a prior to posterior analysis of μ which manifests itself in terms of a scaled distribution, the use of a gamma identity and the use of an exponential change of measure. The rest relies of course on some specific features of the Dirichlet, gamma process calculus. James (2002, sections 5 and 6) demonstrates that using the three general techniques mentioned above, that is elements of what he calls a *Poisson process partition calculus*, that one can obtain a generalization of a Markov-Krein type relationship for all process P defined as in (1) for all random measures μ which are completely random measures or have laws based on a scaling of such processes. In so doing, he shows a natural duality between the posterior distribution of μ and P and the distribution of functionals of P . A key point is that the law of μ can always be represented in terms of a mixture representation derived from its posterior distribution. In principle, this serves to answer a question raised in Tsilevich, Vershik and Yor (2001a). We should say that in retrospect some of those results in James (2002, section 5 and 6) may seem a bit cryptic. The result in Theorem 4.1, without the explicit reference to a beta-gamma process are contained in Proposition 6.1 and Proposition 6.2 of James (2002, section 6). It is of interest to clarify and refine the results in James (2002, section 6) for more general P . In particular, we believe it would be useful to combine these ideas with the related results of Regazzini, Lijoi and Pruenster (2003).

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